RESEARCH ARTICLE

On Modification of Some Estimators using Parameters of Auxiliary Information for the Estimation of the Population Coefficient of Variation

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Abstract

Several studies in the theory of sampling survey have established the fact that the use of auxiliary information at the planning and estimation stages helps in enhancing the efficiency of estimators for estimating population parameters like population mean, population variance, standard deviation etc. as compared to the estimators which use not auxiliary information. In the present study, four estimators for estimating the population coefficient of variation of the study variable using auxiliary information were proposed. The properties (Biases and MSEs) of the proposed estimators were derived up to first order of approximation using Taylor series approach. Numerical analysis was conducted to justify the efficiencies of the proposed estimators and the results revealed that the proposed estimators are more efficient than the existing estimators considered in the study.

Keywords: Auxiliary information; Efficiency; Coefficient of Variation, Mean Square Error; Percentage Relative Efficiency

Introduction

The auxiliary information in sampling theory is used for improved estimation of parameters enhancing the efficiencies of the estimators. The use of auxiliary information is well known to improve the precision of the estimate of the population mean and other parameters of the study variable in survey sampling. Ratio methods of estimation is quite effective when there is high positive correlation between study and auxiliary variables. However, if correlation is negative and very high, the product method of estimation can be employed effectively. Several authors like Ahmed *et al.* (2016), Audu *et al.* (2017), Audu *et al.* (2021), Audu and Singh (2020), Muili *et al.* (2020), Sahai and Ray (1980), Srivastava and Jhajj (1981), Yunusa *et al.* (2021), Adejumobi and Yunusa (2022) have worked extensively in that direction. Authors like Singh and Tailor (2005), Sisodia and Dwivedi (1981), Khoshnevisan *et al.* (2007), Singh *et al.* (2018), Audu and Adewara (2017) utilized coefficient of variation of auxiliary variable in the estimators' formulation and obtained highly efficient estimators. Nevertheless, the investigators did not emphasize the problem of estimation of coefficient of variation. The coefficient of variation is one of the major parameters of population often used in comparing variability measured in different units.

The coefficient of variation is expressed in percentages to indicate the extent of variability percent in the data. In order to have an efficient estimator, we estimate the parameter under study. Recently, we proposed efficient estimators of population coefficient of variation under simple random sampling (SRS) using known auxiliary parameters. These proposed estimators are required to yield efficient estimate of the population coefficient of variation than the existing estimators considered in the study.

Let us consider a finite population $U=\left(U_1,U_2,......,U_N\right)$ of size N consisting of distinct and identifiable units. Let $Y\ and\ X$ denote the study and auxiliary variables and let $Y_i\ and\ X_i$ be their values corresponding to i^{th} unit in the population $\left(i=1,2,.....,N\right)$. For the population observations, means, mean squares and covariance for the study and auxiliary variables, we define as

$$\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i\,,\ \ \bar{X}=\frac{1}{N}\sum_{i=1}^{N}X_i\,,\ S_Y^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^2,\ S_X^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2 \text{ and}$$

$$S_{YX}=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)\left(X_i-\bar{X}\right)$$

For the sample observations, means, mean squares and covariance for the study and auxiliary variables, we define as

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i\,,\ \bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i\,,\ s_y^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2,\ s_x^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2 \text{ and}$$

$$s_{yx}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)\left(x_i-\bar{x}\right).$$

Existing estimators in the literature

The usual unbiased estimator to estimate the population coefficient of variation is given by:

$$t_0=\hat{C}_y=\frac{s_y}{\bar{y}} \tag{1}$$

The mean square error (MSE) expression of the estimator t_0 is given by:

$$MSE\left(t_0\right)=\psi C_y^2\left(C_y^2+0.25\left(\lambda_{04}-1\right)-C_y\lambda_{30}\right) \tag{2}$$

Archana and Rao (2014) proposed ratio-type estimator for the population coefficient of variation, is given by

$$t_{AR}=C_y\left(\frac{\bar{X}}{\bar{x}}\right) \tag{3}$$

The mean square error indication of the estimator t_{AR} is given by:

$$MSE\left(t_{AR}\right)=\psi C_y^2\left(C_y^2+0.25\left(\lambda_{04}-1\right)-C_y\lambda_{30}-C_x\lambda_{21}+0.25\left(\lambda_{04}-1\right)\right) \tag{4}$$

Singh *et al.* (2018) proposed ratio-type, exponential ratio-type and difference-type estimators for coefficient of variation of the study variable Y using mean of the auxiliary variable and are given below with their MSEs as

$$t_1 = \hat{C}_y \left(\frac{\bar{X}}{\bar{x}} \right)^{\alpha} \tag{5}$$

$$t_2 = \hat{C}_y \exp\left\{ \beta \left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \tag{6}$$

$$t_3 = \hat{C}_y + d_1 \left(\bar{X} - \bar{x} \right) \tag{7}$$

$$MSE\left(t_1\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40} - 1}{4} + \alpha^2 C_x^2 - C_y \lambda_{30} + 2\rho_{yx} C_y C_x - \alpha C_x \lambda_{21} \right] \tag{8}$$

$$MSE\left(t_2\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{\beta^2 C_x^2}{4} - C_y \lambda_{30} + \beta \rho_{yx} C_y C_x - \frac{\beta}{2} C_x \lambda_{21} \right] \tag{9}$$

$$MSE\left(t_3\right) = \psi \left[C_y^2 \left(C_y^2 - C_y \lambda_{30} + \frac{\lambda_{40} - 1}{4} \right) + d_1^2 \bar{X}^2 C_x^2 + 2 d_1 \bar{X} \rho_{yx} C_y C_x - d_1 \bar{X} C_x C_y \lambda_{21} \right] \tag{10}$$

where, $\alpha = \dfrac{\lambda_{21} - 2\rho_{yx} C_y}{2C_x}$, $\beta = \dfrac{\lambda_{21} - 2\rho_{yx} C_y}{C_x}$, $d_1 = \dfrac{C_y \lambda_{21} - 2\rho_{yx} C_y^2}{2\bar{X} C_x}$

Singh *et al.* (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_1 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_4^{AM} = \frac{\hat{C}_y}{2} \left[1 + \left(\frac{\bar{X}}{\bar{x}} \right)^{\alpha} \right] \tag{11}$$

$$t_4^{GM} = \hat{C}_y \left[\frac{\bar{X}}{\bar{x}} \right]^{\alpha/2} \tag{12}$$

$$t_4^{HM} = 2\hat{C}_y \left[1 + \left(\frac{\bar{X}}{\bar{x}} \right)^{\alpha} \right]^{-1} \tag{13}$$

$$MSE\left(t_4^r\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40} - 1}{4} + \frac{\alpha^2 C_x^2}{4} - C_y \lambda_{30} + \alpha \rho_{yx} C_y C_x - \frac{\alpha}{2} C_x \lambda_{21} \right] \tag{14}$$

where, $\alpha = \dfrac{\lambda_{21} - 2\rho_{yx} C_y}{2C_x}$, $r = AM, GM \,\&\, HM$

Singh *et al.* (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_5^{AM} = \frac{\hat{C}_y}{2} \left[1 + \exp\left\{ \beta \left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \right\} \right] \tag{15}$$

$$t_5^{GM} = \hat{C}_y \left\{ \frac{\beta}{2}\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \right\} \quad (16)$$

$$t_5^{HM} = 2\hat{C}_y \left[1+\exp\left\{ \beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \right\} \right]^{-1} \quad (17)$$

$$MSE\left(t_5^r\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{\beta^2 C_x^2}{16} - C_y\lambda_{30} + \frac{\beta}{2}\rho_{yx}C_yC_x - \frac{\beta}{4}C_x\lambda_{21} \right] \quad (18)$$

where, $\beta = \dfrac{2\left(\lambda_{21}-2\rho_{yx}C_y\right)}{C_x}$

Singh *et al.* (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_1 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_6^{AM} = \frac{\hat{C}_y}{2}\left[\left(\frac{\bar{X}}{\bar{x}} \right)^{\alpha} + \exp\left\{ \beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \right\} \right] \quad (19)$$

$$t_6^{AM} = \hat{C}_y \left[\left(\frac{\bar{X}}{\bar{x}} \right)^{\alpha/2} + \exp\left\{ \beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \right\} \right] \quad (20)$$

$$t_6^{HM} = 2\hat{C}_y \left[\left(\frac{\bar{x}}{\bar{X}} \right)^{\alpha} + \exp\left\{ -\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \right\} \right]^{-1} \quad (21)$$

$$MSE\left(t_6^r\right) = \psi C_y^2 \left[\begin{array}{l} C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)^2 C_x^2 - C_y\lambda_{30} + \left(\alpha+\frac{\beta}{2}\right)\rho_{yx}C_yC_x \\ -\frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)C_y\lambda_{21} \end{array} \right] \quad (22)$$

where, $\beta = 2\left(\dfrac{\lambda_{21}-2\rho_{yx}C_y}{C_x} - \alpha \right)$

Audu *et al.* (2020) suggested the following two difference cum ratio type estimators of C_y utilizing the known $\bar{X}$ as:

$$t_{a1} = \left[\frac{\hat{C}_y}{2}\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}} \right) + w_1\left(\bar{X}-\bar{x}\right) + w_2\hat{C}_y \right]\left(\frac{\bar{X}}{\bar{x}} \right) \quad (23)$$

$$MSE\left(t_{a1}\right) = C_y^2\left(A + w_1^2B + w_2^2C + 2w_1D - 2w_2E - 2w_1w_2F \right) \quad (24)$$

where,

$$A=\psi\left(C_x^2+C_y^2+2\rho_{yx}C_yC_x-C_x\lambda_{21}-C_y\lambda_{30}+\frac{(\lambda_{40}-1)}{4}\right),\qquad B=\psi\delta^2C_x^2,\qquad \delta=\frac{\bar{X}}{C_y}$$

$$C=1+\psi\left(3C_x^2+3C_y^2+4\rho_{yx}C_yC_x-2C_x\lambda_{21}-2C_y\lambda_{30}\right),\qquad D=\psi\delta\left(C_x^2+\rho_{yx}C_yC_x-\frac{C_x\lambda_{21}}{2}\right)$$

$$E=\psi\left(\frac{3C_x\lambda_{21}}{2}-3\rho_{yx}C_yC_x-\frac{5C_x^2}{2}-2C_y^2+\frac{3C_y\lambda_{30}}{2}+\frac{(\lambda_{40}-1)}{8}\right),\ F=\psi\delta\left(\frac{C_x\lambda_{21}}{2}-\rho_{yx}C_yC_x-2C_x^2\right)$$

Adichwal *et al*. (2016) suggested the following estimator for estimating C_y using the known S_x^2 as,

$$t_7=\delta_1\left[\frac{(1-\eta)s_x^2+\eta S_x^2}{\eta s_x^2+(1-\eta)S_x^2}\right]\hat{C}_y+(1-\delta_1)\left[\frac{\eta s_x^2+(1-\eta)S_x^2}{(1-\eta)s_x^2+\eta S_x^2}\right] \tag{25}$$

Where δ_1 and η are the characterizing constants to be determined such that MSEs of the estimators t_7 is least. The minimum MSEs of the estimator t_7 for the optimum values of these constants is,

$$MSE(t_7)=MSE(t_0)-\frac{1}{4}\psi\frac{\left[(\lambda_{22}-1)-2C_y\lambda_{12}\right]^2}{(\lambda_{04}-1)}C_y^4 \tag{26}$$

Singh *et al*. (2018) proposed ratio-type, exponential ratio-type and difference-type estimators for coefficient of variation of the study variable Y using mean of the auxiliary variable and are given below with their MSEs as

$$t_8=\hat{C}_y\left(\frac{S_x^2}{s_x^2}\right)^{\alpha} \tag{27}$$

$$t_9=\hat{C}_y\exp\left\{\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\} \tag{28}$$

$$t_{10}=\hat{C}_y+d_2\left(S_x^2-s_x^2\right) \tag{29}$$

$$MSE(t_8)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\alpha^2(\lambda_{04}-1)-C_y\lambda_{30}-\alpha(\lambda_{22}-1)+2\alpha C_y\lambda_{12}\right] \tag{30}$$

$$MSE(t_9)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\beta^2(\lambda_{04}-1)}{4}-C_y\lambda_{30}-\beta C_y\lambda_{12}+\frac{\beta}{2}(\lambda_{22}-1)\right] \tag{31}$$

$$MSE(t_{10})=\psi\left[\begin{array}{l}C_y^2\left(C_y^2-C_y\lambda_{30}+\dfrac{\lambda_{40}-1}{4}\right)+d_2^2S_x^4(\lambda_{04}-1)+2C_y^2d_2S_x^2\lambda_{12}\\ -d_2C_yS_x^2(\lambda_{22}-1)\end{array}\right] \tag{32}$$

where, $\alpha=\dfrac{\lambda_{22}-1-2C_y\lambda_{21}}{2(\lambda_{04}-1)}$, $\beta=\dfrac{\lambda_{22}-1-2C_y\lambda_{21}}{(\lambda_{04}-1)}$, $d_2=\dfrac{C_y(\lambda_{22}-1)-2C_y^2\lambda_{12}}{2S_x^2(\lambda_{04}-1)}$

Singh *et al*. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{11}^{AM} = \frac{\hat{C}_y}{2}\left[1+\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}\right] \tag{33}$$

$$t_{11}^{GM} = \hat{C}_y\left(\frac{S_x^2}{s_x^2}\right)^{\alpha/2} \tag{34}$$

$$t_{11}^{HM} = 2\hat{C}_y\left[1+\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}\right]^{-1} \tag{35}$$

$$MSE\left(t_{11}^{r}\right)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\alpha^2\left(\lambda_{04}-1\right)}{4}-C_y\lambda_{30}+\alpha C_y\lambda_{12}-\frac{\alpha}{2}\left(\lambda_{22}-1\right)\right] \tag{36}$$

where, $\alpha=\frac{\lambda_{22}-1-2C_y\lambda_{21}}{\lambda_{04}-1}$, $r = AM, GM \,\&\, HM$

Rajyaguru *et al.* (2002) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_9 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{12}^{AM} = \frac{\hat{C}_y}{2}\left[1+\exp\left\{\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\}\right] \tag{37}$$

$$t_{12}^{GM} = \hat{C}_y\left\{\frac{\beta}{2}\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\} \tag{38}$$

$$t_{12}^{HM} = 2\hat{C}_y\left[1+\exp\left\{-\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\}\right]^{-1} \tag{39}$$

$$MSE\left(t_{12}^{r}\right)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\beta^2\left(\lambda_{04}-1\right)}{16}-C_y\lambda_{30}+\frac{\beta}{2}C_y\lambda_{12}-\frac{\beta}{4}\left(\lambda_{22}-1\right)\right] \tag{40}$$

where, $\beta=\frac{2\left(\lambda_{22}-1\right)-4C_y\lambda_{21}}{\left(\lambda_{04}-1\right)}$, $r = AM, GM \,\&\, HM$

Singh *et al.* (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_8 and t_9 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{13}^{AM} = \frac{\hat{C}_y}{2}\left[\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}+\exp\left\{\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\}\right] \tag{41}$$

$$t_{13}^{AM} = \hat{C}_y\left(\frac{S_x^2}{s_x^2}\right)^{\alpha/2}\exp\left\{\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\} \tag{42}$$

$$t_{13}^{HM} = 2\hat{C}_y\left[\left(\frac{s_x^2}{S_x^2}\right)^{\alpha} + \exp\left\{-\beta\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right\}\right]^{-1} \quad (43)$$

$$MSE\left(t_{13}^{r}\right) = \psi C_y^2 \left[\begin{array}{l} C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{1}{4}\left(\alpha + \frac{\beta}{2}\right)^2\left(\lambda_{40}-1\right) - C_y\lambda_{30} \\ + \frac{1}{2}\left(\alpha + \frac{\beta}{2}\right)\left(\lambda_{22}-1\right) + \left(\alpha + \frac{\beta}{2}\right)C_y\lambda_{12} \end{array}\right] \quad (44)$$

where, $\beta = 2\left(\frac{\left(\lambda_{22}-1\right) - 2C_y\lambda_{12}}{\left(\lambda_{40}-1\right)} - \alpha\right)$

Audu *et al.* (2020) suggested the following two difference cum ratio type estimators of C_y utilizing the known S_x^2 as:

$$t_{a2} = \left[\frac{\hat{C}_y}{2}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2}\right) + w_3\left(S_x^2 - s_x^2\right) + w_4\hat{C}_y\right]\left(\frac{S_x^2}{s_x^2}\right) \quad (45)$$

$$MSE\left(t_{a2}\right) = C_y^2\left(A + w_3^2B_1 + w_4^2C_1 + 2w_3D_1 - 2w_4E_1 - 2w_3w_4F_1\right) \quad (46)$$

where,

$$A_1 = \psi\left(\left(\lambda_{04}-1\right) + C_y^2 + 2C_y\lambda_{12} - C_y\lambda_{30} - \left(\lambda_{22}-1\right) + \frac{\left(\lambda_{40}-1\right)}{4}\right), \quad B_1 = \psi\delta_1^2C_x^2, \quad \delta_1 = \frac{S_x^2}{C_y}$$

$$C_1 = 1 + \psi\left(3\left(\lambda_{04}-1\right) + 3C_y^2 + 4C_y\lambda_{12} - 2\left(\lambda_{22}-1\right) - 2C_y\lambda_{30}\right), \quad D_1 = \psi\delta_1\left(\left(\lambda_{04}-1\right) + C_y\lambda_{12} - \frac{\left(\lambda_{22}-1\right)}{2}\right)$$

$$E_1 = \psi\left(\frac{3\left(\lambda_{22}-1\right)}{2} - 3C_y\lambda_{12} - \frac{5\left(\lambda_{04}-1\right)}{2} - 2C_y^2 + \frac{3C_y\lambda_{30}}{2} + \frac{\left(\lambda_{40}-1\right)}{8}\right)$$

$$F_1 = \psi\delta_1\left(\frac{\left(\lambda_{22}-1\right)}{2} - C_y\lambda_{12} - 2\left(\lambda_{04}-1\right)\right)$$

Yunusa *et al*. (2021) suggested the following log type ratio estimator of C_y utilizing the known S_x^2 as:

$$t_{14} = \hat{C}_y \left(\frac{Ln\left(S_x^2\right)}{Ln\left(s_x^2\right)} \right) \quad (47)$$

The MSE of the estimator t_8, up to the first order of approximation is,

$$MSE\left(t_{14}\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{\lambda_{40}-1}{\left\{Ln\left(S_x^2\right)\right\}^2} - \frac{\left\{\left(\lambda_{22}-1\right) - 2C_y\lambda_{12}\right\}}{Ln\left(S_x^2\right)} - C_y\lambda_{30} \right]$$

(48)

Singh and Kumari (2022) proposed four estimators for coefficient of variation based on information on a single auxiliary variable.

$$t_{p1} = \left[\frac{\hat{C}_y}{2}\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}} \right) + k_1\left(\bar{X} - \bar{x}\right) + k_2\hat{C}_y \right] \left\{ 2 - \left(\frac{\bar{x}}{\bar{X}}\right) \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}} \right) \right\} \quad (49)$$

$$t_{p2} = \left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \right) + k_3\left(\bar{X} - \bar{x}\right) + k_4\hat{C}_y \right] \left\{ 2 - \left(\frac{\bar{x}}{\bar{X}}\right) \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}} \right) \right\} \quad (50)$$

$$t_{p3} = \left[\frac{\hat{C}_y}{2}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2} \right) + k_5\left(S_x^2 - s_x^2\right) + k_6\hat{C}_y \right] \left(2 - \frac{s_x^2}{S_x^2} \right) \quad (51)$$

$$t_{p4} = \left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) + \exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right) \right) + k_7\left(S_x^2 - s_x^2\right) + k_8\hat{C}_y \right] \left(2 - \frac{s_x^2}{S_x^2} \right) \quad (52)$$

$$MSE\left(t_{p1}\right) = C_y^2\left(A_1 + k_1^2B_1 + k_2^2C_1 + 2k_1D_1 - 2k_2E_1 - 2k_1k_2F_1\right) \quad (53)$$

$$MSE\left(t_{p2}\right) = C_y^2\left(A_2 + k_3^2B_2 + k_4^2C_2 + 2k_3D_2 - 2k_4E_2 - 2k_3k_4F_2\right) \quad (54)$$

$$MSE\left(t_{p3}\right) = C_y^2\left(A_3 + k_5^2B_3 + k_6^2C_3 + 2k_5D_3 - 2k_6E_3 - 2k_5k_6F_3\right) \quad (55)$$

$$MSE\left(t_{p3}\right) = C_y^2\left(A_4 + k_7^2B_4 + k_8^2C_4 + 2k_7D_4 - 2k_8E_4 - 2k_7k_8F_4\right) \quad (56)$$

where,

$$A_1=\psi\left(C_y^2+\frac{9C_x^2}{4}+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+3C_{yx}-\frac{3C_x\lambda_{21}}{2}\right),\quad B_1=\psi g^2C_x^2,\quad g=\frac{\bar{X}}{C_y}$$

$$C_1=1+\psi\left(3C_y^2+\frac{3C_x^2}{2}+6C_{yx}-2C_y\lambda_{30}-3C_x\lambda_{21}\right),\quad D_1=\psi g\left(\frac{3C_x^2}{2}+C_{yx}-\frac{C_x\lambda_{21}}{2}\right)$$

$$E_1=\psi\left(\frac{9C_x\lambda_{21}}{4}-\frac{9C_{yx}}{2}-\frac{19C_x^2}{8}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{4}\right),\ F_1=\psi g\left(\frac{C_x\lambda_{21}}{2}-C_{yx}-3C_x^2\right)$$

$$A_2=\psi\left(C_y^2+\frac{9C_x^2}{4}+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+3C_{yx}-\frac{3C_x\lambda_{21}}{2}\right),\quad B_2=\psi g^2C_x^2,\quad g=\frac{\bar{X}}{C_y}$$

$$C_2=1+\psi\left(3C_y^2+\frac{3C_x^2}{2}+6C_{yx}-2C_y\lambda_{30}-3C_x\lambda_{21}\right),\quad D_2=\psi g\left(\frac{3C_x^2}{2}+C_{yx}-\frac{C_x\lambda_{21}}{2}\right)$$

$$E_2=\psi\left(\frac{9C_x\lambda_{21}}{4}-\frac{9C_{yx}}{2}-2C_x^2-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right),\ F_2=\psi g\left(\frac{C_x\lambda_{21}}{2}-C_{yx}-3C_x^2\right)$$

$$A_3=\psi\left(C_y^2+(\lambda_{04}-1)+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+2C_y\lambda_{12}-(\lambda_{22}-1)\right),\quad B_3=\psi g_1^2(\lambda_{04}-1),$$

$$C_3=1+\psi\left(3C_y^2+(\lambda_{04}-1)+4C_y\lambda_{12}-2C_y\lambda_{30}-\frac{3(\lambda_{22}-1)}{2}\right),\ D_3=\psi g_1\left((\lambda_{04}-1)+C_y\lambda_{12}-\frac{(\lambda_{22}-1)}{2}\right)$$

$$E_3=\psi\left(\frac{3(\lambda_{22}-1)}{2}-\frac{3(\lambda_{04}-1)}{2}-3C_y\lambda_{12}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right),\quad g_1=\frac{S_x^2}{C_y}$$

$$F_3=\psi g_1\left(\frac{(\lambda_{22}-1)}{2}-C_y\lambda_{12}-2(\lambda_{04}-1)\right)$$

$$A_4=\psi\left(C_y^2+(\lambda_{04}-1)+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+2C_y\lambda_{12}-(\lambda_{22}-1)\right),\quad B_4=\psi g_1^2(\lambda_{04}-1),$$

$$C_4=1+\psi\left(3C_y^2+(\lambda_{04}-1)+4C_y\lambda_{12}-2C_y\lambda_{30}-\frac{3(\lambda_{22}-1)}{2}\right),\ D_4=\psi g_1\left((\lambda_{04}-1)+C_y\lambda_{12}-\frac{(\lambda_{22}-1)}{2}\right)$$

$$E_4=\psi\left(\frac{3(\lambda_{22}-1)}{2}-\frac{9(\lambda_{04}-1)}{8}-3C_y\lambda_{12}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right),\quad g_1=\frac{S_x^2}{C_y}$$

$$F_4=\psi g_1\left(\frac{(\lambda_{22}-1)}{2}-C_y\lambda_{12}-2(\lambda_{04}-1)\right)$$

Materials And Methods

Proposed Estimators

Having investigated the existing estimators in section 1 (Introduction), we signified four estimators for the coefficient of variation based on information on a single auxiliary variable in three sections.

$$t_{am1} = \left[\frac{\hat{C}_y}{2} \left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}} \right) + u_1 \left(\bar{X} - \bar{x} \right) + u_2 \hat{C}_y \right] \left\{ 2 - \left(\frac{\bar{x}}{\bar{X}} \right) \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right) \right\}^{3/2} \quad (57)$$

$$t_{am2} = \left[\frac{\hat{C}_y}{2} \left(\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) + \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right) \right) + u_3 \left(\bar{X} - \bar{x} \right) + u_4 \hat{C}_y \right] \left\{ 2 - \left(\frac{\bar{x}}{\bar{X}} \right) \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right) \right\}^{3/2} \quad (58)$$

$$t_{am3} = \left[\frac{\hat{C}_y}{2} \left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2} \right) + u_5 \left(S_x^2 - s_x^2 \right) + u_6 \hat{C}_y \right] \left(2 - \frac{s_x^2}{S_x^2} \right)^{1/4} \quad (59)$$

$$t_{am4} = \left[\frac{\hat{C}_y}{2} \left(\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) + \exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2} \right) \right) + u_7 \left(S_x^2 - s_x^2 \right) + u_8 \hat{C}_y \right] \left(2 - \frac{s_x^2}{S_x^2} \right)^{1/4} \quad (60)$$

Properties (Bias and MSE) of $t_{amk},\ k=1,2,3,4$

Now let us define:

$$\varepsilon_0 = \frac{\bar{y}}{\bar{Y}} - 1, \quad \varepsilon_1 = \frac{\bar{x}}{\bar{X}} - 1, \quad \varepsilon_2 = \frac{s_y^2}{S_y^2} - 1, \quad \varepsilon_2 = \frac{s_x^2}{S_x^2} - 1$$

Such that,

$$\left.\begin{array}{l} E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = E(\varepsilon_3) = 0 \\ E(\varepsilon_0^2) = \psi C_y^2,\ E(\varepsilon_1^2) = \psi C_x^2,\ \ E(\varepsilon_2^2) = \psi(\lambda_{40} - 1),\ \ E(\varepsilon_3^2) = \psi(\lambda_{04} - 1) \\ E(\varepsilon_0 \varepsilon_1) = \psi C_{yx} = \psi \rho_{yx} C_y C_x,\ E(\varepsilon_0 \varepsilon_2) = \psi C_y \lambda_{30},\ \ E(\varepsilon_0 \varepsilon_3) = \psi C_y \lambda_{12}, \\ E(\varepsilon_1 \varepsilon_2) = \psi C_x \lambda_{21},\ \ E(\varepsilon_1 \varepsilon_3) = \psi C_x \lambda_{03},\ \ E(\varepsilon_2 \varepsilon_3) = \psi(\lambda_{22} - 1) \end{array}\right\} \quad (61)$$

where,

$\psi = n^{-1} - N^{-1},\ C_y = \frac{S_y}{\bar{Y}},\ C_x = \frac{S_x}{\bar{X}}$ are the population coefficient of variation for the study variable Y and auxiliary variable X. More so, ρ_{yx} denotes the correlation coefficient between X and Y.

$$\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2} \mu_{02}^{s/2}}, \qquad \mu_{rs} = \frac{1}{N-1} \sum_{i=1}^{N} \left(Y_i - \bar{Y} \right)^r \left(X_i - \bar{X} \right)^s$$

The estimators t_{amk}, $k = 1, 2, 3, 4$ in terms of errors can be expressed and simplified respectively, we have

$$t_{am1}=\begin{bmatrix}\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}}\left(\dfrac{\bar{X}}{\bar{X}\left(1+\varepsilon_1\right)}+\dfrac{\bar{X}\left(1+\varepsilon_1\right)}{\bar{X}}\right)\\+u_1\left(\bar{X}-\bar{X}\left(1+\varepsilon_1\right)\right)+\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{bmatrix}\left(2-\frac{\bar{X}\left(1+\varepsilon_1\right)}{\bar{X}}\exp\left(\frac{\bar{X}\left(1+\varepsilon_1\right)-\bar{X}}{\bar{X}\left(1+\varepsilon_1\right)+\bar{X}}\right)\right)^{3/2} \tag{62}$$

$$t_{am2}=\begin{bmatrix}\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}\left(1+\varepsilon_0\right)}\begin{bmatrix}\exp\left(\dfrac{\bar{X}-\bar{X}\left(1+\varepsilon_1\right)}{\bar{X}+\bar{X}\left(1+\varepsilon_1\right)}\right)\\+\exp\left(\dfrac{\bar{X}\left(1+\varepsilon_1\right)-\bar{X}}{\bar{X}\left(1+\varepsilon_1\right)+\bar{X}}\right)\end{bmatrix}\\+u_3\left(\bar{X}-\bar{X}\left(1+\varepsilon_1\right)\right)+u_4\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{bmatrix}\left(2-\frac{\bar{X}\left(1+\varepsilon_1\right)}{\bar{X}}\exp\left(\frac{\bar{X}\left(1+\varepsilon_1\right)-\bar{X}}{\bar{X}\left(1+\varepsilon_1\right)+\bar{X}}\right)\right)^{3/2} \tag{63}$$

$$t_{am3}=\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}\left(1+\varepsilon_0\right)}\begin{bmatrix}\left(\dfrac{S_x^2}{S_x^2\left(1+\varepsilon_3\right)}+\dfrac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)\\+u_5\left(S_x^2-S_x^2\left(1+\varepsilon_3\right)\right)+u_6\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{bmatrix}\left(2-\frac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)^{1/4} \tag{64}$$

$$t_{am4}=\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}\left(1+\varepsilon_0\right)}\begin{bmatrix}\exp\left(\dfrac{S_x^2-S_x^2\left(1+\varepsilon_3\right)}{S_x^2+S_x^2\left(1+\varepsilon_3\right)}\right)+\exp\left(\dfrac{S_x^2\left(1+\varepsilon_3\right)-S_x^2}{S_x^2\left(1+\varepsilon_3\right)+S_x^2}\right)\\+u_7\left(S_x^2-S_x^2\left(1+\varepsilon_3\right)\right)+u_8\dfrac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{bmatrix}\left(2-\frac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)^{1/4} \tag{65}$$

$$t_{am1}=C_y\begin{bmatrix}\left(1+\dfrac{\varepsilon_2}{2}-\dfrac{9\varepsilon_1}{4}+\dfrac{25\varepsilon_1^2}{32}-\varepsilon_0+\dfrac{9\varepsilon_0\varepsilon_1}{4}+\varepsilon_0^2-\dfrac{9\varepsilon_1\varepsilon_2}{8}-\dfrac{\varepsilon_0\varepsilon_1}{2}-\dfrac{\varepsilon_2^2}{8}\right)+u_1\dfrac{\bar{X}}{C_y}\left(\varepsilon_1-\dfrac{9\varepsilon_1^2}{4}\right)\\+u_2\left(1-\dfrac{9\varepsilon_1}{2}+\dfrac{9\varepsilon_1^2}{32}-\varepsilon_0+\dfrac{9\varepsilon_0\varepsilon_1}{4}+\varepsilon_0^2+\dfrac{\varepsilon_2}{2}-\dfrac{9\varepsilon_1\varepsilon_2}{8}-\dfrac{\varepsilon_0\varepsilon_2}{2}-\dfrac{\varepsilon_2^2}{8}\right)\end{bmatrix} \tag{66}$$

$$t_{am2}=C_y\begin{bmatrix}\left(1+\dfrac{\varepsilon_2}{2}-\dfrac{9\varepsilon_1}{4}+\dfrac{13\varepsilon_1^2}{32}-\varepsilon_0+\dfrac{9\varepsilon_0\varepsilon_1}{4}+\varepsilon_0^2-\dfrac{9\varepsilon_1\varepsilon_2}{8}-\dfrac{\varepsilon_0\varepsilon_2}{2}-\dfrac{\varepsilon_2^2}{8}\right)-u_3\dfrac{\bar{X}}{C_y}\left(\varepsilon_1-\dfrac{9\varepsilon_1^2}{4}\right)\\+u_4\left(1-\dfrac{9\varepsilon_1}{2}+\dfrac{9\varepsilon_1^2}{32}-\varepsilon_0+\dfrac{9\varepsilon_0\varepsilon_1}{4}+\varepsilon_0^2+\dfrac{\varepsilon_2}{2}-\dfrac{9\varepsilon_1\varepsilon_2}{8}-\dfrac{\varepsilon_0\varepsilon_2}{2}-\dfrac{\varepsilon_2^2}{8}\right)\end{bmatrix} \tag{67}$$

$$t_{am3}=C_y\begin{bmatrix}\left(1+\dfrac{\varepsilon_2}{2}-\dfrac{\varepsilon_3}{4}+\dfrac{13\varepsilon_3^2}{32}-\varepsilon_0+\dfrac{\varepsilon_0\varepsilon_3}{4}+\varepsilon_0^2-\dfrac{\varepsilon_2\varepsilon_3}{8}-\dfrac{\varepsilon_0\varepsilon_2}{2}-\dfrac{\varepsilon_2^2}{8}\right)+u_5\dfrac{S_x^2}{C_y}\left(\varepsilon_3-\dfrac{\varepsilon_3^2}{4}\right)\\+u_6\left(1-\dfrac{\varepsilon_3}{2}-\dfrac{3\varepsilon_3^2}{32}-\varepsilon_0+\dfrac{\varepsilon_1\varepsilon_3}{4}+\varepsilon_0^2+\dfrac{\varepsilon_2}{2}-\dfrac{\varepsilon_2\varepsilon_3}{8}-\dfrac{\varepsilon_0\varepsilon_2}{2}-\dfrac{\varepsilon_2^2}{8}\right)\end{bmatrix} \tag{68}$$

$$t_{am4}=C_y\left[\begin{array}{l}\left(1+\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{4}+\frac{\varepsilon_3^2}{32}-\varepsilon_0+\frac{\varepsilon_0\varepsilon_3}{4}+\varepsilon_0^2-\frac{\varepsilon_2\varepsilon_3}{8}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)-u_7\frac{S_x^2}{C_y}\left(\varepsilon_3-\frac{\varepsilon_3^2}{4}\right)\\+u_8\left(1-\frac{\varepsilon_3}{4}-\frac{3\varepsilon_3^2}{32}-\varepsilon_0+\frac{\varepsilon_0\varepsilon_3}{4}+\varepsilon_0^2+\frac{\varepsilon_2}{2}-\frac{\varepsilon_2\varepsilon_3}{8}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \tag{69}$$

Subtracting C_y from both sides and take expectations of (66), (67), (68) and (69) to obtain the biases of the estimators to first order of approximations, we have:

$$Bias\left(t_{am1}\right)=C_y\left[\begin{array}{l}\psi\left(\begin{array}{l}\frac{25}{32}C_x^2+\frac{9}{4}\rho_{yx}C_yC_x+C_y^2-\frac{9}{8}C_x\lambda_{21}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)-u_1\frac{\bar{X}}{C_y}\left(-\frac{9}{4}C_x^2\right)\\+u_2\left(1+\psi\left(\begin{array}{l}\frac{9}{32}C_x^2+\frac{9}{4}\rho_{yx}C_yC_x+C_y^2-\frac{9}{8}C_x\lambda_{21}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)\right)\end{array}\right] \tag{70}$$

$$Bias\left(t_{am2}\right)=C_y\left[\begin{array}{l}\psi\left(\begin{array}{l}\frac{13}{32}C_x^2+\frac{9}{4}\rho_{yx}C_yC_x+C_y^2-\frac{9}{8}C_x\lambda_{21}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)-u_3\frac{\bar{X}}{C_y}\left(-\frac{9}{4}C_x^2\right)\\+u_4\left(1+\psi\left(\begin{array}{l}\frac{9}{32}C_x^2+\frac{9}{4}\rho_{yx}C_yC_x+C_y^2-\frac{9}{8}C_x\lambda_{21}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)\right)\end{array}\right] \tag{71}$$

$$Bias\left(t_{am3}\right)=C_y\left[\begin{array}{l}\psi\left(\begin{array}{l}\frac{13}{32}\left(\lambda_{40}-1\right)+\frac{C_y\lambda_{12}}{4}+C_y^2-\frac{\left(\lambda_{22}-1\right)}{8}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)-u_5\frac{S_x^2}{C_y}\left(-\frac{\left(\lambda_{40}-1\right)}{4}\right)\\+u_6\left(1+\psi\left(\begin{array}{l}C_y^2-\frac{3\left(\lambda_{04}-1\right)}{32}+\frac{C_y\lambda_{12}}{4}-\frac{\left(\lambda_{22}-1\right)}{8}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)\right)\end{array}\right] \tag{72}$$

$$Bias\left(t_{am4}\right)=C_y\left[\begin{array}{l}\psi\left(\begin{array}{l}\frac{\left(\lambda_{04}-1\right)}{32}+\frac{C_y\lambda_{12}}{4}+C_y^2-\frac{\left(\lambda_{22}-1\right)}{8}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)-u_7\frac{S_x^2}{C_y}\left(-\frac{\left(\lambda_{04}-1\right)}{4}\right)\\+u_8\left(1+\psi\left(\begin{array}{l}C_y^2-\frac{3\left(\lambda_{04}-1\right)}{32}+\frac{C_y\lambda_{12}}{4}-\frac{\left(\lambda_{22}-1\right)}{8}\\-\frac{C_y\lambda_{30}}{2}-\frac{\left(\lambda_{40}-1\right)}{8}\end{array}\right)\right)\end{array}\right] \tag{73}$$

Subtracting C_y from both sides of (66), (67), (68) and (69), squaring and taking expectations, we have the MSEs of the suggested estimators as:

$$MSE\left(t_{am1}\right)=C_y^2\left(A_1^*+u_1^2B_1^*+u_2^2C_1^*-2u_1D_1^*+2u_2E_1^*-2u_1u_2F_1^*\right) \tag{72}$$

$$MSE\left(t_{am2}\right)=C_y^2\left(A_2^*+u_3^2B_2^*+u_4^2C_2^*-2u_3D_2^*+2u_4E_2^*-2u_3u_4F_2^*\right) \tag{73}$$

$$MSE\left(t_{am3}\right)=C_y^2\left(A_3^*+u_5^2B_3^*+u_6^2C_3^*-2u_5D_3^*+2u_6E_3^*-2u_5u_6F_3^*\right) \tag{74}$$

$$MSE\left(t_{am4}\right)=C_y^2\left(A_4^*+u_7^2B_4^*+u_8^2C_4^*-2u_7D_4^*+2u_8E_4^*-2u_7u_8F_4^*\right) \tag{75}$$

where,

$$A_1^*=\psi\left[\frac{\left(\lambda_{40}-1\right)}{4}-\frac{9}{4}C_x\lambda_{21}-C_y\lambda_{30}+\frac{81}{16}C_x^2+\frac{9}{2}C_{yx}+C_y^2\right],\quad B_1^*=\psi\left[\frac{\bar{X}}{C_y}\right]^2C_x^2$$

$$C_1^*=1+\psi\left[3C_y^2+\frac{45}{8}C_x^2+9C_{yx}-2C_y\lambda_{30}-\frac{9}{2}C_x\lambda_{21}\right],\ D_1^*=\psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2}-\frac{9}{4}C_x^2-C_{yx}\right]$$

$$E_1^*=\psi\left[2C_y^2+\frac{187}{32}C_x^2+\frac{27}{4}C_{yx}-\frac{27}{8}C_x\lambda_{21}-\frac{3C_y\lambda_{30}}{2}+\frac{\left(\lambda_{40}-1\right)}{8}\right],F_1^*=\psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2}-C_{yx}-\frac{9}{2}C_x^2\right]$$

$$A_2^*=\psi\left[\frac{\left(\lambda_{40}-1\right)}{4}-\frac{9}{4}C_x\lambda_{21}-C_y\lambda_{30}+\frac{81}{16}C_x^2+\frac{9}{2}C_{yx}+C_y^2\right],\quad B_2^*=\psi\left[\frac{\bar{X}}{C_y}\right]^2C_x^2$$

$$C_2^*=1+\psi\left[3C_y^2+\frac{45}{8}C_x^2+9C_{yx}-2C_y\lambda_{30}-\frac{9}{2}C_x\lambda_{21}\right],\quad D_2^*=\psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2}-\frac{9}{4}C_x^2-C_{yx}\right]$$

$$E_2^*=\psi\left[2C_y^2+\frac{175}{32}C_x^2+\frac{27}{4}C_{yx}-\frac{27}{8}C_x\lambda_{21}-\frac{3C_y\lambda_{30}}{2}+\frac{\left(\lambda_{40}-1\right)}{8}\right],$$

$$F_2^*=\psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2}-C_{yx}-\frac{9}{2}C_x^2\right]$$

$$A_3^* = \psi\left[\frac{(\lambda_{40}-1)}{4} - \frac{(\lambda_{22}-1)}{4} - C_y\lambda_{30} + \frac{(\lambda_{04}-1)}{16} + \frac{C_y\lambda_{12}}{2} + C_y^2\right], \quad B_3^* = \psi\left[\frac{S_x^2}{C_y}\right]^2(\lambda_{04}-1)$$

$$C_3^* = 1+\psi\left[C_y^2 + \frac{(\lambda_{04}-1)}{8} + C_y\lambda_{12} - \frac{(\lambda_{22}-1)}{2} - 2C_y\lambda_{30}\right], \quad D_3^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - \frac{(\lambda_{04}-1)}{4}\right]$$

$$E_3^* = \psi\left[2C_y^2 - \frac{3(\lambda_{22}-1)}{8} - \frac{3C_y\lambda_{30}}{2} + \frac{15(\lambda_{04}-1)}{32} - \frac{3C_y\lambda_{12}}{4} + \frac{(\lambda_{40}-1)}{8}\right],$$

$$F_3^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - C_y\lambda_{12} - \frac{(\lambda_{04}-1)}{2}\right]$$

$$A_4^* = \psi\left[\frac{(\lambda_{40}-1)}{4} - \frac{(\lambda_{22}-1)}{4} - C_y\lambda_{30} + \frac{(\lambda_{04}-1)}{16} + 2C_y\lambda_{12} + C_y^2\right], \quad B_4^* = \psi\left[\frac{S_x^2}{C_y}\right]^2(\lambda_{04}-1)$$

$$C_4^* = 1+\psi\left[3C_y^2 + \frac{(\lambda_{04}-1)}{8} + C_y\lambda_{12} - \frac{(\lambda_{22}-1)}{2} - 2C_y\lambda_{30}\right], \quad D_4^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - \frac{(\lambda_{04}-1)}{4}\right]$$

$$E_4^* = \psi\left[2C_y^2 - \frac{3(\lambda_{22}-1)}{8} - \frac{3C_y\lambda_{30}}{2} + \frac{3(\lambda_{04}-1)}{32} - \frac{3C_y\lambda_{12}}{4} + \frac{(\lambda_{40}-1)}{8}\right],$$

$$F_4^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - C_y\lambda_{12} - \frac{(\lambda_{04}-1)}{2}\right]$$

Differentiating (72) partially with respect u_1 and u_2, equate to zero and solve for u_1 and u_2 simultaneously, we acquired $u_1 = \frac{E_1^*F_1^* - C_1^*D_1^*}{F_1^{*2} - B_1^*C_1^*}$ and $u_2 = \frac{B_1^*E_1^* - D_1^*F_1^*}{F_1^{*2} - B_1^*C_1^*}$. Substituting the results in (72), we acquired the minimum mean square error of t_{am1} implied by $MSE(t_{am1})_{\min}$;

$$MSE(t_{am1})_{\min} = C_y^2\left[A_1^* + \frac{C_1^*D_1^{*2} - B_1^*E_1^{*2} - 2D_1^*E_1^*F_1^*}{F_1^{*2} - B_1^*C_1^*}\right] \tag{76}$$

Differentiating (73) partially with respect u_3 and u_4, equate to zero and solve for u_3 and u_4 simultaneously, we acquired $u_3 = \frac{E_2^*F_2^* - C_2^*D_2^*}{F_2^{*2} - B_2^*C_2^*}$ and $u_4 = \frac{B_2^*E_2^* - D_2^*F_2^*}{F_2^{*2} - B_2^*C_2^*}$. Substituting the results in (73), we acquired the minimum mean square error of t_{am2} implied by $MSE(t_{am2})_{\min}$;

$$MSE(t_{am2})_{\min} = C_y^2\left[A_2^* + \frac{C_2^*D_2^{*2} - B_2^*E_2^{*2} - 2D_2^*E_2^*F_2^*}{F_2^{*2} - B_2^*C_2^*}\right] \tag{77}$$

Differentiating (74) partially with respect u_5 and u_6, equate to zero and solve for u_5 and u_6 simultaneously, we acquired $u_5=\frac{E_3^*F_3^*-C_3^*D_3^*}{F_3^{*2}-B_3^*C_3^*}$ and $u_6=\frac{B_3^*E_3^*-D_3^*F_3^*}{F_3^{*2}-B_3^*C_3^*}$. Substituting the results in (74), we acquired the minimum mean square error of t_{am3} implied by $MSE\left(t_{am3}\right)_{\min}$;

$$MSE\left(t_{am3}\right)_{\min}=C_y^2\left[A_3^*+\frac{C_3^*D_3^{*2}-B_3^*E_3^{*2}-2D_3^*E_3^*F_3^*}{F_3^{*2}-B_3^*C_3^*}\right] \quad (78)$$

Differentiating (75) partially with respect u_7 and u_8, equate to zero and solve for u_7 and u_8 simultaneously, we acquired
$u_7=\frac{E_4^*F_4^*-C_4^*D_4^*}{F_4^{*2}-B_4^*C_4^*}$ and $u_8=\frac{B_4^*E_4^*-D_4^*F_4^*}{F_4^{*2}-B_4^*C_4^*}$. Substituting the results in (75), we acquired the minimum mean square error of t_{am4} implied by $MSE\left(t_{am4}\right)_{\min}$;

$$MSE\left(t_{am4}\right)_{\min}=C_y^2\left[A_4^*+\frac{C_4^*D_4^{*2}-B_4^*E_4^{*2}-2D_4^*E_4^*F_4^*}{F_4^{*2}-B_4^*C_4^*}\right] \quad (79)$$

Results and Discussion

In this section, we performed numerical analysis to clarify the performance of suggested estimators t_{amk}, $k=1,2,3,4$ with respect to some existing estimators using two (2) datasets.

Dataset 1: Murthy (1967)

X: Area under wheat in 1963 Y: Area under wheat in 1964

$N=34,\ n=15,\ \bar{X}=208.88,\ \bar{Y}=199.44,\ C_x=0.72,\ C_y=0.75,\ \rho=0.98,\ \lambda_{21}=1.0045,$
$\lambda_{12}=0.9406,\ \lambda_{40}=3.6161,\ \lambda_{04}=2.8266,\ \lambda_{30}=1.1128,\ \lambda_{03}=0.9206,\ \lambda_{22}=3.0133$

Dataset 2: Singh (2003)

X: Number of fish caught in year 1993 Y: Number of fish caught in year 1995

$N=69,\ n=40,\ \bar{X}=4591.07,\ \bar{Y}=4514.89,\ C_x=1.38,\ C_y=1.35,\ \rho=0.96,\ \lambda_{21}=2.19,$
$\lambda_{12}=2.30,\ \lambda_{40}=7.66,\ \lambda_{04}=9.84,\ \lambda_{30}=1.11,\ \lambda_{03}=2.52,\ \lambda_{22}=8.19$

Table 1: MSEs and PREs of the proposed and existing estimators considered in the study

Estimators	Dataset 1		Dataset 2	
	MSE	PRE	MSE	PRE
Auxiliary Information: $\bar{X}$, $\bar{x}$				
t_0	0.008003575	100	0.03808821	100
t_{AR}	0.027115658	29.47	0.7645918	49.82
t_1	0.006868341	116.53	0.03731461	102.07
t_2	0.006868341	116.53	0.03731461	102.07
t_3	0.006868341	116.53	0.03731461	102.07
t_4^r	0.006868341	116.53	0.03731461	102.07
t_5^r	0.006868341	116.53	0.03731461	102.07
t_6^r	0.006868341	116.53	0.03731461	102.07
t_{a1}	0.006737495	118.79	0.03404568	111.87
t_{p1}	0.006033	132.66	0.036522	104.29
t_{p2}	0.005659	141.43	0.035795	106.41
Proposed estimators $(t_{amk},\ k=1,2)$				
t_{am1}	0.002181943	366.81	0.02755	138.25
t_{am2}	0.001261989	634.20	0.025071	151.92
Auxiliary Information: S_x^2, s_x^2				
t_7	0.00696301	114.94	0.037568	101.38
t_8	0.006962763	114.95	0.037568156	101.38
t_9	0.006962763	114.95	0.037568156	101.38
t_{10}	0.006962763	114.95	0.037568156	101.38
t_{11}^r	0.006962763	114.95	0.037568156	101.38
t_{12}^r	0.006962763	114.95	0.037568156	101.38
t_{13}^r	0.006962763	114.95	0.037568156	101.38
t_{14}	0.00712551	112.32	0.0375686	101.38
t_{a2}	0.006013652	133.09	0.02810758	135.51
t_{p3}	0.006417	124.72	0.125079	30.45
t_{p4}	0.004996	160.19	0.12893	29.54
Proposed estimators $(t_{amk},\ k=3,4)$				
t_{am3}	0.002944	271.86	0.030917	123.30

t_{am4}	0.003682	217.37	0.037597	101.37

The formula for Percentage Relative Efficiency (PRE) is given as:

$$PRE(estimators) = \frac{MSE(t_0)}{MSE(estimator)} \times 100$$

Table 1 indicate the mean square errors and percentage relative efficiencies of the suggested and other existing related estimators considered in the study using two datasets. Results acquired from each type revealed that suggested estimators under each type have minimum MSEs and higher PREs compared to other competing existing estimators. The suggested estimators are more efficient than their counterparts and have higher chances to produce estimates closer to the true values of means for any population of interest.

Conclusion

In this study we have suggested four exponential type estimators for estimating the population coefficient of variation of the study variable Y using the information of auxiliary variables. The properties of the suggested estimators were derived up to first order of approximation using Taylor series techniques. From the empirical study, the results show that the proposed estimator is an improvement on the existing estimators considered in this study. Hence we recommend that the proposed estimators should be used in practice.

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